Definitions and key facts for section 2.3

The invertible matrix theorem
Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either *all* true, or *all* false.

1. A is an invertible matrix.
2. A is row equivalent to the $n \times n$ identity matrix.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an $n \times n$ matrix C such that $CA = I$.
11. There is an $n \times n$ matrix D such that $AD = I$.
12. A^T is an invertible matrix.
13. The columns of A form a basis of $\mathbb{R}^n$.
14. Col $A = \mathbb{R}^n$
15. dim Col $A = n$
16. rank $A = n$
17. Nul $A = \{\mathbf{0}\}$
18. dim Nul $A = 0$
19. The number 0 is *not* an eigenvalue of A.
20. The determinant of A is *not* zero.
21. $(\text{Col } A)^{|} = \{\mathbf{0}\}$
22. $(\text{Nul } A)^{|} = \mathbb{R}^n$
23. $(\text{Row } A)^{|} = \mathbb{R}^n$
24. A has n nonzero singular values

Time will require us to exclude a few of these statements from discussion. In particular, statements 21-24 will not be covered in class and statements 14-18 will only be covered if time permits.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \text{ and } T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

One can show that S is in this case unique and thus we may call it the **inverse** of T.

Fact: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A. Then T is invertible if and only if A is an invertible matrix. Moreover, if S is the inverse of T, then its standard matrix is A^{-1}.